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Large deviations for a Burgers'-type SPDE

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Abstract

We prove a large deviation principle for a class of semilinear stochastic partial differential equations driven by the space–time white noise. This class of equation contains as special cases Burgers equation and the parabolic SPDEs perturbed by the space–time white noise.

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1. Introduction

We consider a family of equations indexed by $\varepsilon > 0$ introduced by Gyöngy (1998):

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial^2 u^\varepsilon}{\partial^2 x}(t,x) + \sqrt{\varepsilon}\sigma(t,x,u^\varepsilon(t,x))\frac{\partial^2 W}{\partial t\,\partial x}(t,x) + \frac{\partial}{\partial x}g(t,x,u^\varepsilon(t,x)) + f(t,x,u^\varepsilon(t,x)), \tag{1.1}$$

with Dirichlet's boundary conditions $u(t,0) = u(t,1) = 0$ for $t \in [0,T]$, and initial condition $u(0,x) = u_0(x)$ for $x \in [0,1]$. W denotes the Brownian sheet defined on a probability space $(\Omega,\mathscr{F},P)$. We assume that σ is bounded and Lipschitz, g (resp. f) has a quadratic (resp. linear) growth. Furthermore, f and g are locally Lipschitz.

If $g(t,x,r) = \frac{1}{2}r^2$, $f = 0$, (1.1) is the stochastic Burgers equation studied by Da Prato et al. (1994) if $\sigma = 1$, and by Da Prato and Gatarek (1995) for more general σ. If $g = 0$, (1.1) is the classical parabolic SPDE, which has been intensively studied.

The aim of this paper is to prove a large deviation principle (LDP) for the law of the solutions u^ε. In the case of parabolic SPDEs, Sowers (1992) proved an LDP in the set of α-Hölder continuous functions for $\alpha < \frac{1}{4}$, when all coefficients are bounded and σ is bounded away from 0. The crucial part of his proof consists in proving an exponential inequality for stochastic integrals, in Hölder norm, which depends on a lemma by Garsia (1972). Such an exponential inequality and the transfer principle

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(Azencott, 1980; Priouret, 1982) allow to weaken the assumption on σ. This line of proof has been used by Chenal and Millet (1997) to prove a uniform LDP in $C^{\alpha,2\alpha}([0,T]\times[0,1])$, $\alpha<\frac{1}{4}$ and by Rovira and Sanz-Solé (1996) to prove an LDP for non-linear hyperbolic SPDE on the set of real continuous functions. By the same method, Rovira and Sanz-Solé (1997) prove an LDP for Volterra's equation in the plane, and Nualart and Rovira (1998) in the multidimensional case.

In this paper, we use a similar method in a slightly different setting. Indeed, Gyöngy has proved that if $u_0\in L^p([0,1])$, $p\geqslant 2$ then trajectories of u^ε have a modification that belongs to $C([0,T];L^p[0,1])$, so that the functions are no longer Hölder continuous. This requires to prove that exponential inequalities are slightly different from the known ones. The paper is organized as follows. In Section 2, we state precise assumptions and the main result. In Section 3, we prove the continuity of the skeleton Z used to describe the rate function. The proof is close to that in Chenal and Millet (1997), but there are two differences: on the one hand, we work here with the $L^p([0,1])$-norm and on the other, the drift contains a quadratic term, which lacks the Lipschitz property. Section 4 is devoted to the proof of the Freidlin–Wentzell inequality, which is based on an exponential inequality for stochastic integrals (generalizing that of Sowers, 1992; Chenal and Millet, 1997). The Appendix is devoted to technical results about the Green kernel.

2. The main LDP

Let us recall the assumptions made by Gyöngy (1998):

(H1) u_0 is a function in $L^p([0,1])$ with $p\geqslant 2$.
The functions f, g et σ are Borel functions on $\mathbb{R}^+\times[0,1]\times\mathbb{R}$ which satisfy the following:

(H2) The function σ is uniformly continuous in the two first variables, uniformly in the third one. Furthermore, σ is bounded and globally Lipschitz in the third variable.

(H3) There exists a constant L such that for $(t,x,p,q)\in[0,T]\times[0,1]\times\mathbb{R}^2$,

$$|f(t,x,p)-f(t,x,q)|+|g(t,x,p)-g(t,x,q)|\leqslant L(1+|p|+|q|)|p-q|. \tag{2.1}$$

(H4) The function g is of the form

$$g(t,x,r)=g_1(t,x,r)+g_2(t,r), \tag{2.2}$$

where g_1 and g_2 are Borel functions such that there is a constant $K>0$ for which for all $(t,x,r)\in[0,T]\times[0,1]\times\mathbb{R}$,

$$|g_1(t,x,r)|\leqslant K(1+|r|),\quad |g_2(t,r)|\leqslant K(1+|r|^2). \tag{2.3}$$

(H5) f satisfies the linear growth condition:

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}|f(t,x,r)|\leqslant K(1+|r|).$$

In Theorem 2.1, Gyöngy (1998) proves for any $\varepsilon > 0$, the existence and uniqueness of the process u^ε solution of (1.1), which admits a modification in $C([0,T], L^p[0,1])$.

As shown in Gyöngy (1998, Proposition 3.5), the following evolution formulation is equivalent with the weak form of (1.1):

$$\begin{aligned} u^\varepsilon(t,x) = & \int_0^1 G_t(x,y)u_0(y)\,\mathrm{d}y + \sqrt{\varepsilon}\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(s,y,u^\varepsilon(s,y))W(\,\mathrm{d}y,\,\mathrm{d}s) \\ & - \int_0^t\int_0^1 \partial_y G_{t-s}(x,y)g(s,y,u^\varepsilon(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ & + \int_0^t\int_0^1 G_{t-s}(x,y)f(s,y,u^\varepsilon(s,y))\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

The function $G_t(.,.)$ is the Green kernel associated with the heat operator $\partial/\partial t - \partial^2/\partial^2 x$ with Dirichlet's boundary conditions (cf. Section A.2 for the exact expression of G_t). We set $G_t u_0(x) = \int_0^1 G_t(x,y)u_0(y)\,\mathrm{d}y$. Notice that for any $p \geqslant 2$, $G_t u_0(.) \in C([0,T], L^p[0,1])$ while if $p > 2$, $u^\varepsilon - Gu_0$ belongs to $C^{\alpha,2\alpha}([0,T]\times[0,1])$ for $0 < \alpha < \frac{1}{2}(\frac{1}{2} - 1/p)$ (cf. Lemmas A.1 and A.2).

Let

$$\mathscr{H} = \left\{ h(t,x) = \int_0^t\int_0^x \dot{h}(u,z)\,\mathrm{d}u\,\mathrm{d}z;\ \dot{h} \in L^2([0,T]\times[0,1]) \right\}$$

be the Cameron–Martin space endowed with the norm $\|h\|_{\mathscr{H}} = (\int_0^T\int_0^1 \dot{h}^2(u,z)\,\mathrm{d}u\,\mathrm{d}z)^{1/2}$.

Given $h \in \mathscr{H}$, consider the PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}(t,x) + \sigma(t,x,u(t,x))\dot{h}(t,x) + \frac{\partial g}{\partial x}(t,x,u(t,x)) + f(t,x,u(t,x)) \tag{2.5}$$

with Dirichlet's boundary conditions $u(t,0) = u(t,1) = 0$ and initial condition $u(0,x) = u_0(x)$.

Theorem 2.1. *Let u_0, f, g and σ satisfy* (H1)–(H5); *for every $h \in \mathscr{H}$ Eq.* (2.5) *admits a unique solution denoted by $Z(h)$ which belongs to* $C([0,T]; L^p[0,1])$.

The proof which is omitted, is similar to that of Theorem 2.1 of Gyöngy (1998) but in a deterministic framework, the stochastic integral being replaced by an integral with respect to $\dot{h}$.

The main result of this paper is the following:

Theorem 2.2. *Let σ, g, f and u_0 satisfy conditions* (H1)–(H5); *then the law of the solution* (u^ε) *of the SPDE* (2.4) *satisfies on* $C([0,T], L^p[0,1])$ *a large deviation principle with the good rate function*:

$$I(f) = \begin{cases} \inf\{\frac{1}{2}\|h\|^2_{\mathscr{H}};\ Z(h) = f\} & \text{if } f \in \mathrm{Im}(Z), \\ +\infty & \text{if } f \notin \mathrm{Im}(Z). \end{cases} \tag{2.6}$$

Notice that if we strenghten (H1) and assume $p > 2$, the random variable $v^\varepsilon(t,x) = u^\varepsilon(t,x) - G_t u_0(x)$ satisfies a stronger LDP in the space of Hölder-continuous functions $C^{\alpha,2\alpha}([0,T]\times[0,1])$ for $0<\alpha<\frac{1}{2}(\frac{1}{2}-1/p)$. Let $Z'(h)$ be the skeleton associated with v^ε, i.e., the solution of

$$\begin{aligned} Z'(h)(t,x) = & \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(s,y,G_s u_0(y) + Z'(h)(s,y))\dot{h}(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ & - \int_0^t \int_0^1 \partial_y G_{t-s}(x,y) g(s,y,G_s u_0(y) + Z'(h)(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ & + \int_0^t \int_0^1 G_{t-s}(x,y)[f(s,y,G_s u_0(y) + Z'(h)(s,y))]\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Then $Z'(h)$ belongs to $C^{\alpha,2\alpha}([0,T]\times[0,1])$, and arguments similar to those in Sections 3 and 4 show that v^ε satisfies an LDP in $C^{\alpha,2\alpha}([0,T]\times[0,1])$ with the good rate function

$$J(f) = \inf\left\{\frac{1}{2}\|h\|^2_{\mathscr{H}};\ Z'(h) = f\right\},$$

with the convention $\inf\emptyset = +\infty$. The contraction principle immediately implies that if $u_0 \in L^p[0,1]$, $p \geqslant 2$, u^ε satisfies an LDP in $C([0,T], L^p[0,1])$, while if $p>2$ and $\alpha < \frac{1}{2} - 1/p$, $u_0 \in C^{2\alpha}([0,1])$, u^ε satisfies an LDP in $C^{\alpha,2\alpha}([0,T]\times[0,1])$ with the good rate function I.

To prove Theorem 2.2, according to Doss and Priouret (1982) and Azencott (1980), we only have to check

- *For any $a>0$, the restriction $Z : (\{I \leqslant a\}, \|.\|_\infty) \to C([0,T], L^p[0,1])$ is continuous.*
- *Freidlin–Wentzell inequality*: For every $\eta>0$, $R>0$, $h\in\mathscr{H}$ there exists $\delta>0$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P\left(\sup_{t\in[0,T]} \|u^\varepsilon(t,.) - Z(h)(t,.)\|_p \geqslant \eta;\ \|W-h\|_\infty < \delta\right) \leqslant -R. \quad (2.7)$$

3. Regularity of the skeleton

The following lemma is crucial to overcome the fact that the coefficients only have local Lipschitz properties.

Lemma 3.1. *Assume* (H1)–(H5), *then for every* $a \in [0,\infty[$,

$$\sup_{\|h\|_{\mathscr{H}} \leqslant a}\ \sup_{t\in[0,T]} \|Z(h)(t,.)\|_p = C_a < \infty. \quad (3.1)$$

The proof which is similar to that of Lemma 3.1 of Da Prato et al. (1994) and Theorem 2.1 of Gyöngy (1998) is omitted.

Proposition 3.2. *Suppose* (H1)–(H5), *let* Z *denote the skeleton defined in Theorem* 2.1. *Then for every* $a>0$, *the map*

$$Z:\{\|h\|_{\mathscr{H}}\leqslant a\}\to C([0,T],L^p([0,1])),$$
$$h\mapsto Z(h)$$

is continuous when the level set $\{\|h\|_{\mathscr{H}}\leqslant a\}$ *is endowed with the topology of uniform convergence on* $[0,T]\times[0,1]$.

Proof. Let k and h be such that $\|h\|_{\mathscr{H}}\vee\|k\|_{\mathscr{H}}\leqslant a$; then

$$|Z(h)(t,x)-Z(k)(t,x)|\leqslant A(t,x)+B(t,x)+C(t,x), \tag{3.2}$$

where

$$A(t,x)=\int_0^t\int_0^1|\partial_y G_{t-s}(x,y)|\,|g(s,y,Z(h)(s,y))-g(s,y,Z(k)(s,y))|\,\mathrm{d}y\,\mathrm{d}s,$$

$$B(t,x)=\int_0^t\int_0^1|G_{t-s}(x,y)|\,|f(s,y,Z(h)(s,y))-f(s,y,Z(k)(s,y))|\,\mathrm{d}y\,\mathrm{d}s,$$

$$C(t,x)=\Bigg|\int_0^t\int_0^1 G_{t-s}(x,y)[\sigma(s,y,Z(h)(s,y))\dot{h}(s,y)$$
$$-\sigma(s,y,Z(k)(s,y))\dot{k}(s,y)]\,\mathrm{d}y\,\mathrm{d}s\Bigg|.$$

The function $(s,y)\mapsto(g(s,y,Z(h)(s,y))-g(s,y,Z(k)(s,y)))$ belongs to $L^\infty([0,T];L^{p/2}[0,1])$; thus (A.1) applied with $q=p/2$, $\rho=p$, $\gamma=\infty$, Schwarz's inequality and (H3) imply

$$\|A(t,.)\|_p\leqslant C\int_0^t(t-s)^{-(1/2p)-(1/2)}\|Z(h)(s,.)+Z(k)(s,.)+1\|_p$$
$$\times\|Z(h)(s,.)-Z(k)(s,.)\|_p\,\mathrm{d}s.$$

Finally, (3.1) and Schwarz's inequality yield

$$\|A(t,.)\|_p^2\leqslant C_a\int_0^t(t-s)^{-(1/2p)-(1/2)}\sup_{r\leqslant s}\|Z(h)(r,.)-Z(k)(r,.)\|_p^2\,\mathrm{d}s. \tag{3.3}$$

A similar (easier) computation shows that

$$\|B(t,.)\|_p^2\leqslant C_a\int_0^t(t-s)^{-(1/2p)-(1/2)}\sup_{r\leqslant s}\|Z(h)(r,.)-Z(k)(r,.)\|_p^2\,\mathrm{d}s. \tag{3.4}$$

The estimate of the last term is more difficult. We have to discretize the Green kernel and the skeleton to prove an upper estimate in terms of the norm $\|h-k\|_\infty$. Thus as in Chenal and Millet (1997), we set

$$\begin{aligned}t_n&=\sup\{2^{-n}(k-1)T\colon 2^{-n}kT\leqslant t\}\vee 0,\\ z_n&=\sup\{2^{-n}k\colon 2^{-n}k\leqslant z\}\end{aligned} \tag{3.5}$$

and

$$\underline{t-s_n}=\begin{cases}t-s_n & \text{if } t\geqslant 2^{-n}T,\\ 2^{-n}T & \text{if } t<2^{-n}T.\end{cases}$$

and let $G_0u_0(z)=u_0(z)$. For every $n\geqslant 1$, set

$$Z_n(h)(t,x)=\begin{cases} 2^n\displaystyle\int_{x_n}^{x_n+2^{-n}} u_0(z)\,\mathrm{d}z & \text{if } 0\leqslant t<T2^{1-n},\\ Z(h)(t_n,x_n) & \text{if } T2^{1-n}\leqslant t<T, \end{cases} \tag{3.6}$$

then

$$C(t,x)\leqslant \sum_{i=1}^{4} C_i(t,x), \tag{3.7}$$

with:

$$C_1(t,x)=\int_0^t\int_0^1 |G_{t-s}(x,z)|\,|\sigma(s,z,Z(k)(s,z))-\sigma(s,z,Z(h)(s,z))|\,|\dot{k}(s,z)|\,\mathrm{d}s\,\mathrm{d}z,$$

$$C_2(t,x)=\int_0^t\int_0^1 |G_{t-s}(x,z)|\,|\sigma(s,z,Z(h)(s,z))-\sigma(s_n,z_n,Z_n(h)(s,z))|$$
$$\times(|\dot{k}(s,z)|+|\dot{h}(s,z)|)\,\mathrm{d}s\,\mathrm{d}z,$$

$$C_3(t,x)=\int_0^t\int_0^1 |G_{t-s}(x,z)-G_{\underline{t-s_n}}(x,z_n)|\times|\sigma(s_n,z_n,Z_n(h)(s,z))|$$
$$\times[|\dot{h}(s,z)|+|\dot{k}(s,z)|]\,\mathrm{d}s\,\mathrm{d}z,$$

$$C_4(t,x)=\left|\int_0^t\int_0^1 G_{\underline{t-s_n}}(x,z_n)\sigma(s_n,z_n,Z_n(h)(s,z))(\dot{h}-\dot{k})(s,z)\,\mathrm{d}s\,\mathrm{d}z\right|.$$

3.1. Estimation of C_1

By Schwarz's inequality, the Lipschitz property of σ, and (A.1) with $\rho=q=p/2$ and $\gamma=\infty$, we deduce

$$|C_1(t,x)|^2\leqslant \|k\|_{\mathscr{H}}^2\int_0^t\int_0^1 |G_{t-s}^2(x,z)||Z(h)(s,z)-Z(k)(s,z)|^2\,\mathrm{d}s\,\mathrm{d}z.$$

$$\leqslant C_a\int_0^t (t-s)^{-1/2}\sup_{r\leqslant s}\|Z(h)(r,.)-Z(k)(r,.)\|_p^2\,\mathrm{d}s. \tag{3.8}$$

3.2. Estimation of C_2

We first prove that Z_n converge to Z; more precisely.

Lemma 3.1. *Suppose* (H1)–(H5); *for every* $a\in[0,\infty[$,

$$\lim_{n\to+\infty}\ \sup_{\|h\|_{\mathscr{H}}\leqslant a}\ \sup_{t\in[0,T]}\|Z(h)(t,.)-Z_n(h)(t,.)\|_p=0. \tag{3.9}$$

Proof. We decompose $Z(h)$, as follows:

$$Z(h)(t,x)=\sum_{i=1}^{4} Z^{(i)}(t,x), \tag{3.10}$$

where

$$Z^{(1)}(t,x) = G_t u_0(x),$$

$$Z^{(2)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) f(s,y,Z(h)(s,y))\,\mathrm{d}y\,\mathrm{d}s,$$

$$Z^{(3)}(t,x) = -\int_0^t \int_0^1 \partial_y G_{t-s}(x,y) g(s,y,Z(h)(s,y))\,\mathrm{d}y\,\mathrm{d}s,$$

$$Z^{(4)}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(s,y,Z(h)(s,y))\dot{h}(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

Let us study the most critical part containing the initial condition. By Lemma A.2, the function $(t \mapsto G_t u_0(.))$ belongs to $C([0,T];L^p[0,1])$; it is also uniformly continuous in time and:

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \|G_{t_n}u_0(.) - G_t u_0(.)\|_p = 0. \tag{3.11}$$

For the space increments, if $t \leqslant T2^{1-n}$, then $Z_n(h)(t,x) = 2^n \int_{x_n}^{x_n+2^{-n}} u_0(z)\,\mathrm{d}z$ is the conditional expectation of u_0 given the σ-field $\mathscr{G}_n = \sigma([i/2^n,(i+1)/2^n[;0\leqslant i \leqslant 2^n)$ for the Lebesgue measure. Therefore, (3.11) and the equality $G_{t_n}u_0(.) = u_0(.)$ for $t < T2^{1-n}$ imply

$$\lim_{n\to\infty} \sup_{t<T2^{1-n}} \|Z_n(h)(t,.) - Z^{(1)}(t,.)\|_p = 0. \tag{3.12}$$

Let $t \geqslant T2^{1-n}$; then $t_n \geqslant T2^{-n}$. Let q be the conjugate exponent of p; Hölder's inequality, then inequalities (A.8) and (A.9) imply for $0<\lambda<1$:

$$\begin{aligned}
&\int_0^1 |G_{t_n}u_0(z_n) - G_{t_n}u_0(z)|^p\,\mathrm{d}z \\
&\leqslant \int_0^1 \left| \int_0^1 (G_{t_n}(x,z_n) - G_{t_n}(x,z))u_0(x)\,\mathrm{d}x \right|^p \mathrm{d}z \\
&\leqslant \|u_0\|_p^p \int_0^1 \left[\int_0^1 |G_{t_n}(x,z_n) - G_{t_n}(x,z)|^q\,\mathrm{d}x \right]^{p/q} \mathrm{d}z \\
&\leqslant C\|u_0\|_p^p \int_0^1 |z-z_n|^{p\lambda} t_n^{-p\lambda} \left[\int_0^1 |G_{t_n}(z_n,x)|^{q(1-\lambda)} + |G_{t_n}(z,x)|^{q(1-\lambda)}\,\mathrm{d}x \right]^{p/q} \mathrm{d}z \\
&\leqslant C2^{-p\lambda n}\|u_0\|_p^p t_n^{-p\lambda} t_n^{p/2q - p(1-\lambda)/2} \leqslant C\|u_0\|_p^p 2^{-np/2(\lambda-1/p)}.
\end{aligned}$$

Therefore, if $\lambda > 1/p$ we obtain that

$$\lim_{n\to+\infty} \sup_{\|h\|_{\mathscr{H}}\leqslant a} \sup_{t\in[T2^{1-n},T]} \int_0^1 |Z^{(1)}(t_n,x) - Z^{(1)}(t_n,x_n)|^p\,\mathrm{d}x = 0. \tag{3.13}$$

Using (A.2) with $\rho = q = p$, $\alpha_1 < \frac{1}{2}$ (resp $\rho = p$ and $q = p/2$, $\alpha_2 < \frac{1}{2} - 1/2p$) we deduce

$$\|Z^{(2)}(h)(t,.) - Z^{(2)}(h)(t_n,.)\|_p \leqslant C_a 2^{-n\alpha}, \tag{3.14}$$

$$\|Z^{(3)}(h)(t,.) - Z^{(3)}(h)(t_n,.)\|_p \leqslant C_a 2^{-n\alpha}. \tag{3.15}$$

For the last term, we use Schwarz's inequality and then inequalities (A.6) and (A.7) to obtain for every x

$$|Z^{(4)}(t,x) - Z^{(4)}(t_n,x)| \leqslant C_a |t - t_n|^{1/4} \leqslant C_a 2^{-n/4}.$$

For the space increment, we repeat the same calculations using inequalities (A.3), (A.5) and (A.7).

Inequalities (3.12)–(3.15) and the fact that, for $t < T2^{1-n}$ and $i \in \{2,3,4\}$, $Z^{(i)}(h)(t_n,x) = 0$, conclude the proof of the lemma. □

The function σ is uniformly continuous, bounded and globally Lipschitz in the third variable; therefore, we have

$$\lim_{n\to\infty} \sup_{\|h\|_{\mathscr{H}} \leqslant a} \sup_{s\in[0,T]} \int_0^1 |\sigma(s,z,Z(h)(s,z)) - \sigma(s_n,z_n,Z_n(h)(s,z))|^p \, dz = 0. \tag{3.16}$$

Finally, (3.16) and (A.1) applied with $q = \rho = p$ and $\gamma = +\infty$ yield

$$\lim_{n\to\infty} \sup_{\|h\|_{\mathscr{H}} \leqslant a} \sup_{t\in[0,T]} \|C_2(t,.)\|_p^2 = 0. \tag{3.17}$$

3.3. Estimation of C_3

Schwarz's inequality, the boundedness of σ and inequalities (A.5) and (A.6) imply

$$\begin{aligned} C_3(t,x) &\leqslant \|\sigma\|_\infty \times C_a \times \left[\int_0^t \int_0^1 |G_{t-s}(x,z) - G_{\underline{t-s_n}}(x,z_n)|^2 \, dz \, ds\right]^{1/2} \\ &\leqslant \|\sigma\|_\infty \times C_a \times 2^{-n/4}. \end{aligned} \tag{3.18}$$

3.4. Estimation of C_4

This term will be estimated in terms of $\|h - k\|_\infty$; indeed $Z_n(h)$ is constant on each rectangle $[\mathrm{i}T2^{-n}, (\mathrm{i}+1)T2^{-n}[\times [\mathrm{j}2^{-n}, (\mathrm{j}+1)2^{-n}[$, so that

$$\begin{aligned} C_4(t,x) \leqslant \sum_{j=0}^{2^n-1} &\left| G_{2^{-n}T}\left(x, \frac{\mathrm{j}}{2^n}\right)\right| \left|\sigma\left(0, \frac{\mathrm{j}}{2^n}, Z_n(h)\left(0, \frac{\mathrm{j}}{2^n}\right)\right)\right| \\ &\times \left|(h-k)\left([0,t] \times \left[\frac{\mathrm{j}}{2^n}, \frac{\mathrm{j}+1}{2^n}\right]\right)\right| 1_{\{t<2^{-n}T\}} \end{aligned}$$

$$+\sum_{i=0}^{2^n-2}\sum_{j=0}^{2^n-1}\left|G_{t-i/2^nT}\left(x,\frac{j}{2^n}\right)\right|\left|\sigma\left(\frac{iT}{2^n},\frac{j}{2^n},Z_n(h)\left(\frac{iT}{2^n},\frac{j}{2^n}\right)\right)\right|$$

$$\times\left|(h-k)\left(\left[\frac{(i+1)T}{2^n}\wedge t,\frac{(i+2)T}{2^n}\wedge t\right]\times\left[\frac{j}{2^n},\frac{j+1}{2^n}\right]\right)\right|1_{\{t\geqslant 2^{-n}T\}}.$$

Estimation (A.4) of the Green kernel yields for $t\in[0,T]$,

$$\sup_{t\in[0,T]}\|C_4(t,.)\|_p^2\leqslant\|\sigma\|_\infty^2\|h-k\|_\infty^2\times 2^{5n}. \tag{3.19}$$

3.5. Continuity of the function Z

Inequalities (3.2)–(3.4), (3.8),(3.17)–(3.19) and Gronwall's lemma yield the existence of a positive sequence $x(n)$ with $\lim_n x(n)=0$, such that for $\|h\|_{\mathscr{H}}\vee\|k\|_{\mathscr{H}}\leqslant a$,

$$\sup_{s\leqslant t}\|Z(h)(s,.)-Z(k)(s,.)\|_p^2\leqslant C_a x(n)+\|\sigma\|_\infty^2\|h-k\|_\infty^2 2^{5n}. \tag{3.20}$$

Fix $\eta>0$; choose n such that $x(n)\leqslant\eta/2C_a$, then choose $\alpha>0$ such that $2^{5n}\alpha^2<\eta/2$; (3.20) implies that if $\|h\|_{\mathscr{H}}\vee\|k\|_{\mathscr{H}}\leqslant a$,

$$\sup_{\|h-k\|_\infty\leqslant\alpha}\ \sup_{t\in[0,T]}\|Z(h)(t,x)-Z(k)(t,x)\|_p^2\leqslant\eta,$$

which concludes the proof of the proposition. □

4. Freidlin–Wentzell's inequality

The argument, similar to the corresponding one in Chenal and Millet (1997), is based on exponential inequalities for the stochastic integral. Notice that here the time and space variable appear with very different norms, which forces to show new inequalities.

Set

$$\tilde W(t,x)=W(t,x)-\frac{h(t,x)}{\sqrt{\varepsilon}} \tag{4.1}$$

and

$$\frac{d\tilde P}{dP}=\exp\left(\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_0^T\dot h(t,x)\,dW(t,x)-\frac{1}{2\varepsilon}\int_0^1\int_0^T\dot h^2(t,x)\,dt\,dx\right). \tag{4.2}$$

By Girsanov's Theorem, under $\tilde P$, the process $\tilde W$ is a Brownian sheet and $P\circ(u^\varepsilon)^{-1}=\tilde P\circ(\tilde u^\varepsilon)^{-1}$, where for $J^\varepsilon(t,x)=\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(y,s,\tilde u^\varepsilon(s,y))\tilde W(dy,ds)$,

$$\begin{aligned}\tilde u^\varepsilon(t,x)=G_tu_0(x)&+\int_0^t\int_0^1 G_{t-s}(x,y)[f(s,y,\tilde u^\varepsilon(s,y))\\&+\sigma(s,y,\tilde u^\varepsilon(s,y))\dot h(s,y)]\,dy\,ds\\&-\int_0^t\int_0^1\partial_yG_{t-s}(x,y)g(s,y,\tilde u^\varepsilon(s,y))\,dy\,ds+\sqrt{\varepsilon}J^\varepsilon(t,x).\end{aligned} \tag{4.3}$$

A standard stopping argument and Gronwall's lemma (based on estimates similar to those made for $A(t,x)$, $B(t,x)$ and $C_1(t,x)$ in the previous section), shows that the proof of the Friedlin–Wentzell inequality (2.7) reduces to check that for every $R>0$, $\eta>0$, $a>0$, there is $\delta>0$ such that if $\|h\|_{\mathscr{H}}\leqslant a$:

$$\limsup_{\varepsilon\to 0}\ \varepsilon\ln\tilde{P}\left(\sup_{t\leqslant\tau_\varepsilon}\|\sqrt{\varepsilon}J_\varepsilon(t,.)\|_p>\eta;\|\sqrt{\varepsilon}\tilde{W}\|_\infty<\delta\right)\leqslant -R, \tag{4.4}$$

where τ_ε is a stopping time such that

$$\sup_{t\leqslant\tau_\varepsilon}\|\tilde{u}_\varepsilon(t,.)\|_p=M<\infty. \tag{4.5}$$

4.1. Exponential inequalities in $C([0,T];L^p[0,1])$

We define a Hölder seminorm of order v, denoted by $[.]_v$ by

$$[\Phi]_v=\sup\left\{\frac{|\Phi(t,x)-\Phi(s,y)|}{(|t-s|^2+|x-y|^4)^{\frac{v}{2}}};(t,x),(s,y)\in[0,T]\times[0,1],(t,x)\neq(s,y)\right\}.$$

We need a result similar to that in Sowers (1992), Chenal and Millet (1997), Rovira and Sanz-Solé (1997) and Nualart and Rovira (1997), but different since we have to deal with the $L^q([0,1])$ norm instead of $L^\infty([0,1])$ for the space variable.

Lemma 4.1. *Let p' et q' be conjugate exponents, such that $1\leqslant q'<\infty$.*
Let $F:[0,T]\times[0,1]\times[0,T]\times[0,1]\to\mathbb{R}$ be a function such that for some $v_0>0$,

$$\left(\int_0^T\int_0^1|F(t,x,u,z)-F(s,y,u,z)|^{2q'}\,\mathrm{d}u\,\mathrm{d}z\right)^{1/q'}\leqslant C_F(|t-s|^2+|x-y|^4)^{v_0} \tag{4.6}$$

and let Y be a process adapted to the filtration $\mathscr{F}_t=\sigma(\tilde{W}_{s,x},s\leqslant t,x\in[0,1])$, such that

$$\|Y^2\|_{L^{p'}([0,T]\times[0,1])}\leqslant K_Y<\infty\quad a.s. \tag{4.7}$$

Set

$$I(t,x)=\int_0^t\int_0^1F(t,x,u,z)Y(u,z)\tilde{W}(\mathrm{d}z,\mathrm{d}u),$$

then for all $v<v_0$, there exist $C(v,v_0)>0$ and $K(v,v_0)>0$, such that if $M>K(v,v_0)C(v,v_0)C_F\sqrt{K_Y}$,

$$\tilde{P}([I]_v\geqslant M)\leqslant(\sqrt{2}T^2+1)\exp\left(\frac{-M^2}{4K_YC_FC^2(v,v_0)}\right). \tag{4.8}$$

Moreover, there exist $C(v_0)>0$ and $K(v_0)>0$, such that if $M>K(v_0)C(v_0)C_F\sqrt{K_Y}$,

$$\tilde{P}\left(\sup_{t\in[0,T]}\|I(t,.)\|_p\geqslant M\right)\leqslant(\sqrt{2}T^2+1)\exp\left(\frac{-M^2}{4K_YC_FC^2(v_0)}\right). \tag{4.9}$$

Proof. To prove inequality (4.8), set $\mu=(\nu+\nu_0)/2$,

$$e((t,x),(s,y))=\sqrt{(t-s)^2+(y-s)^4}\times\sqrt{T/(1+T^2)},$$

$$p(u)=|u|^{\mu}\sqrt{C_F}(1+T^2)^{(\nu-\nu_0)/2}\left(\frac{1+T^2}{T}\right)^{\mu/2}.$$

We introduce the process

$$M_r=\int_0^r\int_0^1[F(t,x,u,z)-F(s,y,u,z)]\times\frac{Y(u,z)}{p[e((t,x),(s,y))]\sqrt{K_Y}}\tilde{W}(\mathrm{d}u,\mathrm{d}z),$$

since Y is $\mathscr{F}_t$-adapted, the process M is a local martingale. The Hölder inequality and inequalities (4.6) and (4.7) imply that the quadratic variation $\langle M\rangle_r\leqslant 1$. Hence $(M_r,0\leqslant r\leqslant T)$ is a continuous martingale and there is a Brownian motion B such as $M_r=B_{\langle M\rangle_r}$ in law. Then, using Lemma 1 of Garsia (1972), we deduce (4.8) as one in the proof of Proposition A.1 of Sowers (1992). To prove (4.9), we proceed in a similar way. □

As usual, the proof of (4.4) is similar to that of Proposition 3.2; it depends on a discretization of the integrand. For every integer $n\geqslant 1$, set

$$\tilde{u}_n^{\varepsilon}(t,x)=\begin{cases}2^n\displaystyle\int_{x_n}^{x_n+2^n}u_0(z)\,\mathrm{d}z & \text{if } t<T2^{1-n},\\ \tilde{u}^{\varepsilon}(t_n,x_n) & \text{if } t\geqslant T2^{1-n}\end{cases}\tag{4.10}$$

and for $t<\tau_\varepsilon$, decompose $J^\varepsilon(t,x)$ as

$$J^{\varepsilon}(t,x)=J_n^{\varepsilon}(t,x)+K_n^{\varepsilon}(t,x)+L_n^{\varepsilon}(t,x),\tag{4.11}$$

where

$$J_n^{\varepsilon}(t,x)=\int_0^{t\wedge\tau_\varepsilon}\int_0^1 G_{\underline{t-s_n}}(x,z_n)\sigma(s_n,z_n,\tilde{u}_n^{\varepsilon}(s,z))\tilde{W}(\mathrm{d}z,\mathrm{d}s),$$

$$K_n^{\varepsilon}(t,x)=\int_0^{t\wedge\tau_\varepsilon}\int_0^1[G_{t-s}(x,z)-G_{\underline{t-s_n}}(x,z_n)]\sigma(s_n,z_n,\tilde{u}_n^{\varepsilon}(s,z))\tilde{W}(\mathrm{d}z,\mathrm{d}s),$$

$$L_n^{\varepsilon}(t,x)=\int_0^{t\wedge\tau_\varepsilon}\int_0^1 G_{t-s}(x,z)[\sigma(s,z,\tilde{u}^{\varepsilon}(s,z))-\sigma(s_n,z_n,\tilde{u}_n^{\varepsilon}(s,z))]\tilde{W}(\mathrm{d}z,\mathrm{d}s).$$

Let us check that for fixed $R>0$ and $\eta>0$, there exists n_0 such that if $n\geqslant n_0$,

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\tilde{P}\left(\sqrt{\varepsilon}\sup_{t\leqslant T}\|K_n^{\varepsilon}(t,.)\|_p\geqslant\eta\right)\leqslant -R,\tag{4.12}$$

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\tilde{P}\left(\sqrt{\varepsilon}\sup_{t\leqslant T}\|L_n^{\varepsilon}(t,.)\|_p\geqslant\eta\right)\leqslant -R.\tag{4.13}$$

Finally, we check that for every $n>0$, there exists $\delta>0$ such that

$$\tilde{P}\left(\sup_{t\leqslant\tau_\varepsilon}\sqrt{\varepsilon}\|J_\varepsilon^n(t,.)\|_p>\frac{\eta}{2};\|\sqrt{\varepsilon}\tilde{W}\|_\infty\leqslant\delta\right)=0.\tag{4.14}$$

Inequalities (4.12)–(4.14) immediately yield (4.4).

4.2. Proof of (4.12)

Apply Lemma 4.1 with

$$Y(s,z)=\sigma(s_n,z_n,\tilde{u}_n^{\varepsilon}(s_n,z_n))1_{\{s\leqslant\tau_\varepsilon\}},\quad F(t,x,s,z)=(G_{t-s}(x,z)-G_{\underline{t-s_n}}(x,z_n))\sqrt{\varepsilon},$$

$p'=\infty$, and $q'=1$. In this case $\|Y^2\|_\infty\leqslant\|\sigma\|_\infty^2$; combining inequalities (A.10)–(A.12), we obtain for $0<K<1$,

$$\int_0^t\int_0^1|F(t,x,u,z)-F(s,y,u,z)|^2\,\mathrm{d}u\,\mathrm{d}z$$
$$\leqslant C\varepsilon[|x-y|^{1-K}2^{-nK^2/1+2K}+2^{-nK/K+3}|t-s|^{1/2-K/2}+2^{-nK/2}|t-s|^{1/2-K/2}]. \tag{4.15}$$

Thus, for $K=1-4v_0$, there exist $\beta>0$ and $C_1>0$ such that

$$\int_0^t\int_0^1|F(t,x,u,z)-F(s,y,u,z)|^2\,\mathrm{d}u\,\mathrm{d}z\leqslant C_1\varepsilon(|x-y|^4+|t-s|^2)^{v_0}2^{-n\beta}. \tag{4.16}$$

Choosing n_0 big enough, we have $\eta\geqslant K(v_0)C(v_0)C_1 2^{-n(\beta/2)}$ and $R\leqslant\eta^2/(4\|\sigma\|_\infty^2 2^{-n\beta}C_1C^2(v_0))$ if $n\geqslant n_0$. Then inequality (4.9) implies for $n\geqslant n_0$ and $\varepsilon\in\,]0,1]$:

$$\tilde{P}\left(\sqrt{\varepsilon}\sup_{t\leqslant T}\|K_n^{\varepsilon}(t,.)\|_p\geqslant\eta\right)\leqslant(\sqrt{2}T^2+1)\exp\left(-\frac{R}{\varepsilon}\right), \tag{4.17}$$

which proves (4.12).

4.3. Proof of (4.13)

Set $F(t,x,s,z)=\sqrt{\varepsilon}G_{t-s}(x,z)$, fix q' in $]1,\frac{3}{2}[$, and let p' be the conjugate exponent of q'. By Inequalities (A.5)–(A.8), we have for $v_0=(3-2q')/4$

$$\left(\int_0^T\int_0^1|F(t,x,u,z)-F(s,y,u,z)|^{2q'}\,\mathrm{d}u\,\mathrm{d}z\right)^{1/q'}\leqslant\varepsilon C_G(|t-s|^2+|x-y|^4)^{v_0}. \tag{4.18}$$

Fix $\beta>0$; the uniform continuity of σ on $[0,T]\times[0,1]$ uniformly in the third variable and the Lipshitz property of $\sigma(s,z,.)$ imply the existence of $\delta>0$ and $n_1\geqslant 0$ such that if $n>n_1$ and $\|\xi-\zeta\|_{L^p([0,T]\times[0,1])}\leqslant\delta$:

$$\|\sigma(s,z,\xi(s,z))-\sigma(s_n,z_n,\zeta(s,z))\|_{L^p([0,T]\times[0,1])}\leqslant\beta.$$

Since the function σ is bounded, we deduce the existence of $\delta>0$ and $n_1>0$ such that if $n\geqslant n_1$ and $\|\xi-\zeta\|_{L^p([0,T]\times[0,1])}\leqslant\delta$:

$$\|\,|\sigma(s,z,\xi(s,z))-\sigma(s_n,z_n,\zeta(s,z))|^2\|_{L^{p'}([0,T]\times[0,1])}\leqslant\beta.$$

Set

$$Y_n^{\varepsilon}(s,z) = [\sigma(s,z,\tilde{u}^{\varepsilon}(s,z)) - \sigma(s_n,z_n,\tilde{u}_n^{\varepsilon}(s,z))]1_{\{\sup_{r\leqslant s}\|\tilde{u}_n^{\varepsilon}(r,.)-\tilde{u}^{\varepsilon}(r,.)\|_p\leqslant\delta\}}1_{\{s\leqslant\tau_\varepsilon\}}.$$

Set $I_n^{\varepsilon}(t,x) = \sqrt{\varepsilon}\int_0^t\int_0^1 G_{t-s}(x,z)Y_n^{\varepsilon}(s,z)\tilde{W}(\mathrm{d}s,\mathrm{d}z)$, and apply Lemma 4.1 with q' and p'. There exists $\varepsilon_0 < 1$ and $\beta > 0$ such that for $0<\varepsilon<\varepsilon_0$, $\eta > K(v_0)C(v_0)\varepsilon C_G$ and $\eta^2 \geqslant 4\beta^2 C_G C^2(v_0)R$; then choose n_1 such that for $n\geqslant n_1$, $\|(Y_n^{\varepsilon})^2\|_{p'}\leqslant\beta$; then

$$\tilde{P}\left(\sup_{t\in[0,T]}\|I^{\varepsilon}(t,.)\|_p\geqslant\eta\right)\leqslant(\sqrt{2}T^2+1)\exp\left(\frac{-\eta^2}{4\beta^2\varepsilon C_G C^2(v_0)}\right). \tag{4.19}$$

Thus, taking $\beta>0$ small enough, we deduce that for $n\geqslant n_1$:

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\tilde{P}\left(\sup_{t\in[0,T]}\|I_n^{\varepsilon}(t,.)\|_p\geqslant\eta\right)\leqslant -R. \tag{4.20}$$

This inequality, the definition of Y_n^{ε} and the local property of stochastic integrals imply

$$\tilde{P}\left(\sup_{t\in[0,T]}\|L_n^{\varepsilon}(t,.)\|_p\geqslant\eta;\sup_{t\leqslant\tau_\varepsilon}\|\tilde{u}^{\varepsilon}(t,.)-\tilde{u}_n^{\varepsilon}(t,.)\|_p\leqslant\delta\right)\leqslant\tilde{P}\left(\sup_{t\leqslant T}\|I_n^{\varepsilon}(t,.)\|_p\geqslant\eta\right). \tag{4.21}$$

Inequalities (4.20) and (4.21) reduce the proof of (4.13) to that of the following:

Lemma 4.22. *For every $\delta>0$ there exists $n_2>0$ such that for $n\geqslant n_2$ and for every h with $\|h\|_{\mathscr{H}}\leqslant a$, if $\tilde{u}_n^{\varepsilon}$ is defined by* (4.10)

$$\limsup_{\varepsilon\to 0}\varepsilon\ln\tilde{P}\left(\sup_{t\in[0,\tau_\varepsilon]}\|\tilde{u}^{\varepsilon}(t,.)-\tilde{u}_n^{\varepsilon}(t,.)\|_p>\delta\right)\leqslant -R. \tag{4.22}$$

Proof. The definition of the seminorm $[.]_v$ implies that

$$[J^{\varepsilon}]_v\geqslant\frac{|J^{\varepsilon}(t_n,x_n)-J^{\varepsilon}(t,x)|}{(T^2 2^{-2n}+2^{-4n})^{v/2}}. \tag{4.23}$$

Moreover, for $t\leqslant\tau^{\varepsilon}$, if $\bar{J}^{\varepsilon}(t,x)=J^{\varepsilon}(t,x)1_{\{t\leqslant\tau^{\varepsilon}\}}$,

$$(\bar{J}^{\varepsilon}(t_n,x_n)-\bar{J}^{\varepsilon}(t,x))\sqrt{\varepsilon}=\tilde{u}_n^{\varepsilon}(t,x)-\tilde{u}^{\varepsilon}(t,x)-\Delta_1^{\varepsilon}(t,x)-\Delta_2^{\varepsilon}(t,x)-\Delta_3^{\varepsilon}(t,x), \tag{4.24}$$

with

$$\begin{aligned}\Delta_1^{\varepsilon}(t,x) &= \int_0^{t_n\wedge\tau_\varepsilon}\int_0^1 \partial_y G_{t_n-s}(x_n,y)g(s,y,\tilde{u}^{\varepsilon}(s,y))\,\mathrm{d}y\,\mathrm{d}s\\ &\quad-\int_0^{t\wedge\tau_\varepsilon}\int_0^1 \partial_y G_{t-s}(x,y)g(s,y,\tilde{u}^{\varepsilon}(s,y))\,\mathrm{d}y\,\mathrm{d}s\\ &\quad+\int_0^{t_n\wedge\tau_\varepsilon}\int_0^1 G_{t_n-s}(x_n,y)f(s,y,\tilde{u}^{\varepsilon}(s,y))\,\mathrm{d}y\,\mathrm{d}s\\ &\quad-\int_0^{t\wedge\tau_\varepsilon}\int_0^1 G_{t-s}(x,y)f(s,y,\tilde{u}^{\varepsilon}(s,y))\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

$$\Delta_2^\varepsilon(t,x) = \int_0^{t_n}\int_0^1 G_{t_n-s}(x_n,y)\sigma(s,y,\tilde{u}^\varepsilon(s,y))\dot{h}(s,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$- \int_0^{t\wedge\tau_\varepsilon}\int_0^1 G_{t-s}(x_n,y)\sigma(s,y,\tilde{u}^\varepsilon(s,y))\dot{h}(s,y)\,\mathrm{d}y\,\mathrm{d}s$$

and

$$\Delta_3^\varepsilon(t,x) = \begin{cases} G_{t_n\wedge\tau_\varepsilon}u_0(x_n) - G_t u_0(x) & \text{if } t \geqslant T2^{1-n}, \\ 2^n \displaystyle\int_{x_n}^{x_n+2^n} u_0(z)\,\mathrm{d}z - G_t u_0(x) & \text{if } t < T2^{1-n}. \end{cases}$$

To estimate Δ_1^ε, we use inequalities (A.2) and(A.3) with $\rho = p,\ q = p/2,\ K = 1 - 1/p$ and (4.5) for $\alpha < \frac{1}{2} - 1/2p$; this yields

$$\sup_{t\leqslant\tau_\varepsilon} \|\Delta_1^\varepsilon(t,.)\|_p \leqslant K_a'(|t-t_n|^\alpha + |x-x_n|^{2\alpha}) \leqslant C2^{-n\alpha}. \tag{4.25}$$

To estimate Δ_2^ε, Schwarz's inequality, the boundedness of σ and inequalities (A.5)–(A.7) imply for $\alpha < \frac{1}{4}$:

$$\sup_{t\leqslant\tau_\varepsilon} \|\Delta_2^\varepsilon(t,.)\|_p \leqslant C2^{-n\alpha}. \tag{4.26}$$

Inequalities (3.12) and (3.13) yield

$$\lim_{n\to+\infty}\left[\sup_{T2^{1-n}\leqslant t\leqslant\tau_\varepsilon} \|\Delta_3^\varepsilon(t,.)\|_p + \sup_{0\leqslant t\leqslant T2^{1-n}} \|\tilde{u}_n^\varepsilon(t,.) - G_t u_0(.)\|_p\right] = 0. \tag{4.27}$$

Inequalities (4.24)–(4.27) imply the existence of a sequence $K(n)$ converging to 0, independent of ε, such that

$$\sup_{t\leqslant\tau_\varepsilon} \|\Delta_1^\varepsilon(t,.) + \Delta_2^\varepsilon(t,.) + \Delta_3^\varepsilon(t,.)\|_p \leqslant K(n), \tag{4.28}$$

$$\sup_{t\leqslant T} \sqrt{\varepsilon}\left[\int_0^1 |\bar{J}^\varepsilon(t_n,x_n) - \bar{J}^\varepsilon(t,x)|^p\,\mathrm{d}x\right]^{1/p} \geqslant \sup_{t\leqslant\tau_\varepsilon} \|\tilde{u}_n^\varepsilon(t,.) - \tilde{u}^\varepsilon(t,.)\|_p - CK(n).$$

Choose $n_3 > 0$ such that for $n \geqslant n_3$, $CK(n) \leqslant \delta/2$; for $n \geqslant n_3$, (4.23) and (4.29) imply

$$\tilde{P}\left(\sup_{t\leqslant\tau_\varepsilon} \|\tilde{u}_n^\varepsilon(t,.) - \tilde{u}^\varepsilon(t,.)\|_p \geqslant \delta\right) \leqslant \tilde{P}\left(\sup_{(t,x)\in[0,T]\times[0,1]} |\sqrt{\varepsilon}(\bar{J}^\varepsilon(t,x) - \bar{J}^\varepsilon(t_n,x_n)| \geqslant \delta\right)$$
$$\leqslant \tilde{P}\left(\sqrt{\varepsilon}[\bar{J}^\varepsilon]_v \geqslant \frac{\delta}{2(T^2 2^{-2n} + 2^{-4n})^{v/2}}\right). \tag{4.29}$$

To estimate the right-hand side of (4.29), we use Lemma 4.1 with $q' = 1,\ p' = \infty$,

$$Y(s,z) = \sigma(s,z,\tilde{u}^\varepsilon(s,z))\sqrt{\varepsilon} \quad \text{and} \quad F(t,u,x,z) = G_{t-u}(x,z).$$

For $v < v_0 = \frac{1}{4}$, choose n_4 such that for $n \geqslant n_4$:

$$\delta(T^2 2^{-2n} + 2^{-4n})^{-v/2} > K(v,v_0)C(v,v_0)C_G$$

and

$$\delta^2[16\|\sigma\|_\infty^2 C^2(v,v_0)(T^2 2^{-2n} + 2^{-4n})^v]^{-1} \geqslant R.$$

For $n \geqslant n_4$, $0 < \varepsilon < 1$ we have

$$\tilde{P}\left([\sqrt{\varepsilon}\bar{J}^{\varepsilon}]_{v} \geqslant \frac{\delta}{2(T^2 2^{-2n} + 2^{-4n})^{v/2}}\right)$$

$$\leqslant (\sqrt{2}T^2 + 1)\exp -\left(\frac{\delta^2}{16\|\sigma\|_{\infty}^2 \varepsilon C^2(v, v_0)(T^2 2^{-2n} + 2^{-4n})^{v}}\right). \tag{4.30}$$

Inequalities (4.29) and (4.30) yield

$$\tilde{P}\left(\sup_{t \leqslant \tau_{\varepsilon}} \|\tilde{u}^{\varepsilon}(t,.) - \tilde{u}_n^{\varepsilon}(t,.)\|_p \geqslant \delta\right) \leqslant (\sqrt{2}T^2 + 1)\exp\left(-\frac{R}{\varepsilon}\right), \tag{4.31}$$

for $n \geqslant n_3 \vee n_4$. This concludes the proof of the lemma. □

4.4. Proof of inequality (4.14)

Since $\tilde{u}_n^{\varepsilon}$ is constant on the rectangles $[iT2^{-n}, (i+1)T2^{-n}[\times [j2^{-n}, (j+1)T2^{-n}[$, computations similar to those used to check (3.19) imply

$$\sup_{t \leqslant \tau_{\varepsilon}} \|J_n^{\varepsilon}(t,.)\|_p \leqslant C 2^{5n/2} \|\tilde{W}\|_{\infty}.$$

Thus, for fixed $n > 0$, there exists $\delta > 0$ such that (4.4) holds.

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Appendix A

To make reading easier, we quote here some results of Gyöngy (1998) and Chenal and Millet (1997).

A.1. Regularizing effect of the Green kernel

The following result is Lemma 3.1 of Gyöngy (1998):

Lemma A.1. *Let J be a linear operator defined for all $v \in L^{\infty}([0,T], L^1[0,1])$, $t \in [0,T]$, and $x \in [0,1]$ by*

$$J(v)(t,x) = \int_0^t \int_0^1 H(r,t,x,y)v(r,y)\,\mathrm{d}y\,\mathrm{d}r,$$

with $H(t,s,x,y) = G_{t-s}(x,y)$, $G_{t-s}^2(x,y)$ or $\partial_y G_{t-s}(x,y)$. Then for any $\rho \in [1,\infty]q \in [1,\rho], q < \infty$ such that $K = 1 + 1/\rho - 1/q > 0$, J is a bounded linear operator of $L^{\gamma}([0,T]; L^q[0,1])$ in $C([0,T]; L^{\rho}[0,1])$ for $\gamma > 2K^{-1}$. Moreover, J satisfies the *following inequalities*:

1. *For every* $T \geqslant 0$, *for every* $t \leqslant T$ *and* $\gamma > 2K^{-1}$,

$$\|J(v)(t,.)\|_\rho \leqslant C_1 \int_0^t (t-r)^{(1/2)K-1} \|v(r,.)\|_q \,\mathrm{d}r. \tag{A.1}$$

2. *For every* $T \geqslant 0$, $0 < \alpha < \frac{1}{2}K$, *and* $\gamma > (\frac{1}{2}K - \alpha)^{-1}$ *there is* $C_2 > 0$ *such that for all* $s \leqslant t \leqslant T$

$$\|J(v)(t,.) - J(v)(s,.)\|_\rho \leqslant C_2 (t-s)^\alpha \| 1_{[0,t]}(r) \|v(r,.)\|_{L^q[0,1]} \|_{L^\gamma[0,T]}. \tag{A.2}$$

3. *For every* $T > 0$, $0 < \beta < K$, *and* $\delta > 2(K-\beta)^{-1}$ *there is* $C_3 > 0$ *such that for all* $t \in [0,T]$, $z \in \mathbb{R}$

$$\|J(v)(t,.) - J(v)(t,.+z)\|_\rho \leqslant C_3 . z^\beta \| 1_{[0,t]}(r) \|v(r,.)\|_{L^q[0,1]} \|_{L^\delta[0,T]}. \tag{A.3}$$

A.2. Some inequalities on the Green kernel

Let

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{n=+\infty} \left[\exp\left\{ \frac{-(y-x-2n)^2}{4t} \right\} - \exp\left\{ \frac{-(y+x-2n)^2}{4t} \right\} \right]$$

be the Green kernel associated with $\partial/\partial t - \partial^2/\partial x^2$ and the Dirichlet boundary conditions. The following results are either classical or proved in the appendix of Chenal and Millet (1997).

For all $x, y \in [0,1]$, $0 < s \leqslant t \leqslant T$, we have

$$|G_t(x,y)| \leqslant \frac{C}{\sqrt{t}} \exp\left(-\frac{(y-x)^2}{4t} \right), \tag{A.4}$$

$$\sup_{t\in[0,T]} \int_0^t \int_0^1 |G_u(x,z) - G_u(y,z)|^p \,\mathrm{d}z\,\mathrm{d}u \leqslant C|x-y|^{3-p}, \quad \tfrac{3}{2} < p < 3, \tag{A.5}$$

$$\sup_{x\in[0,1]} \int_0^s \int_0^1 |G_{t-u}(x,z) - G_{s-u}(x,z)|^p \,\mathrm{d}z\,\mathrm{d}u \leqslant C|t-s|^{(3-p)/2}, \quad 1 < p < 3, \tag{A.6}$$

$$\sup_{x\in[0,1]} \int_s^t \int_0^1 |G_u(x,z)|^p \,\mathrm{d}z\,\mathrm{d}u \leqslant C|t-s|^{(3-p)/2}, \quad 1 < p < 3. \tag{A.7}$$

There is a constant $C > 0$ such that for all x, y and $z \in [0,1]$, and $u \in \,]0,T]$

$$|G_u(x,z) - G_u(y,z)| \leqslant C u^{-1} |x-y|. \tag{A.8}$$

For $\beta \geqslant 0$, for all $(t,x) \in \,]0,T] \times [0,1]$,

$$\sup_{x\in[0,1]} \int_0^1 |G_t(x,z)|^\beta \,\mathrm{d}z \leqslant C t^{-\beta/2+1/2}. \tag{A.9}$$

Finally, if $K \in]0,1[$, for all $x, y \in [0,1]$ and $t \in [0,T]$ we have

$$\int_0^t \int_0^1 |G_{t-u}(x,z) - G_{\underline{t-u_n}}(x,z_n) - G_{t-u}(y,z) + G_{\underline{t-u_n}}(y,z_n)|^2 \,\mathrm{d}z\,\mathrm{d}u$$
$$\leqslant C|x-y|^{1-K} 2^{-K^2/(1+2K)n}. \tag{A.10}$$

Let K be in $]0,\frac{1}{2}[$; for all $x \in [0,1]$ and $0 \leqslant s \leqslant t \leqslant T$ we have

$$\int_0^s \int_0^1 |G_{t-u}(x,z) - G_{\underline{t-u_n}}(x,z_n) - G_{s-u}(x,z) + G_{\underline{s-u_n}}(x,z_n)|^2 \,\mathrm{d}z\,\mathrm{d}u$$
$$\leqslant C|t-s|^{1/2-K} 2^{-2Km/(3+2K)}, \tag{A.11}$$

$$\int_s^t \int_0^1 |G_{t-u}(x,z) - G_{\underline{t-u_n}}(x,z_n)|^2 \,\mathrm{d}z\,\mathrm{d}u \leqslant C|t-s|^{1/2-K} 2^{-Kn}. \tag{A.12}$$

The following lemma proves the continuity of the action of G_t on $L^p([0,1])$.

Lemma A.2. *Let u_0 be a function of $L^p[0,1]$, $p \geqslant 2$. Then the application $(t \mapsto G_t u_0)$ is continuous from $[0,1]$ in $L^p[0,1]$; moreover*

$$\sup_{0 \leqslant t \leqslant T} \|G_t u_0\|_p \leqslant C\|u_0\|_p. \tag{A.13}$$

Proof. If the function u_0 is continuous, the proof of this lemma is straightforward. If $u_0 \in L^p[0,1]$, u_0 is the limit in $L^p[0,1]$ of a sequence of continuous functions, and it suffices to prove (A.13). We apply successively Hölder's inequality (A.9) and Fubini's theorem to obtain

$$\begin{aligned}
\|G_t u_0\|_p^p &= \int_0^1 \left[\int_0^1 G_t(x,y) u_0(y)\,\mathrm{d}y\right]^p \mathrm{d}x \\
&\leqslant \int_0^1 \left[\int_0^1 G_t(x,y) u_0(y)^p\,\mathrm{d}y\right] \left[\int_0^1 G_t(x,y)\,\mathrm{d}y\right]^{p-1} \mathrm{d}x \\
&\leqslant K_p \int_0^1 \int_0^1 G_t(x,y) u_0(y)^p\,\mathrm{d}y\,\mathrm{d}x \leqslant K_p \|u_0\|_p^p. \qquad \square
\end{aligned}$$

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